UDC 539.3

ON THE EXISTENCE OF SOLUTIONS OF THE NONLINEAR THEORY OF SHALLOW SHELLS

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The solvability of the equilibrium equations of isotropic, homogeneous, shallow shells of constant thickness (the V. Z. Vlasov modification) has been examined in [1]. The results of [1] are extended herein to the case of orthotropic inhomogeneous, shallow shells of variable thickness with considerably more general boundary conditions. A generalization of the Korn inequality is obtained as an intermediate result. The possibility of applying projection methods is noted. Determination of the generalized solution of the problem, and obtaining the fundamental a priori estimate of the solutions differ somewhat from those in [1].

1. Fundamental relationships. Let the shell middle surface S^* be given by the equation $\mathbf{r} = \mathbf{r}(\alpha, \beta)$, which maps S^* homeomorphically into some domain Ω of the plane of the variables α, β. The following modification of the relations of nonlinear theory is considered for an elastic orthotropic inhomogeneous shallow shell of variable thickness

$$\begin{gathered}
\varepsilon_1 = A^{-1}u_\alpha + A_\beta v (AB)^{-1} + wR_1^{-1} + {}^1\!/_2 w_\alpha^2 A^{-2} \\
\varepsilon_2 = B^{-1}v_\beta + B_\alpha u (AB)^{-1} + wR_2^{-1} + {}^1\!/_2 w_\beta^2 B^{-2} \\
\varepsilon_3 = \frac{A}{B}\left(\frac{u}{A}\right)_\beta + \frac{B}{A}\left(\frac{v}{B}\right)_\alpha + \frac{w_\alpha w_\beta}{AB} - 2\frac{w}{R_{12}} \qquad (1.1) \\
\varkappa_1 = -\frac{1}{A}\left(\frac{w_\alpha}{A}\right)_\alpha - \frac{A_\beta w_\beta}{AB^2}, \qquad \varkappa_2 = -\frac{1}{B}\left(\frac{w_\beta}{B}\right)_\beta - \frac{B_\alpha w_\alpha}{A^2 B} \\
\tau = -(AB)^{-1} w_{\alpha\beta} + B_\alpha w_\beta A^{-1} B^{-2} + A_\beta w_\alpha A^{-2} B^{-1} \\
T_1 = E_{11}\varepsilon_1 + E_{12}\varepsilon_2, \qquad T_2 = E_{21}\varepsilon_1 + E_{22}\varepsilon_2, \qquad S = G_1\varepsilon_3 \\
M_1 = D_1\varkappa_1 + D_{12}\varkappa_2, \qquad M_2 = D_{21}\varkappa_1 + D_2\varkappa_2, \qquad M = D_k\tau \\
E_{ii} = 2hE_i(1 - \mu_1\mu_2)^{-1} \quad (i = 1, 2), \quad E_{12} = \mu_2 E_{11}, \quad E_{21} = \mu_1 E_{22}, \quad G_1 = 2hG \\
3D_i = 2h^3 E_i (1 - \mu_1\mu_2)^{-1} \quad (i = 1, 2), \quad D_{12} = \mu_2 D_1, \quad D_{21} = \mu_1 D_2, \quad 3D_k = 4Gh^3
\end{gathered}$$

The following notation is used in (1.1): u, v, w are displacements of a point of the shell middle surface, A^2, B^2, $2C = 0$ are coefficients of the first quadratic form of the surface, R_1, R_2, R_{12} are the radii of curvature of the middle surface, ε_1, ε_2, ε_3 are the tensile and shear strains, $\varkappa_1$, $\varkappa_2$, τ are the changes in curvature of the shell middle surface, T_1, T_2, S are tangential stress resultants, M_1, M_2, M are the bending and twisting moments. E_1, E_2, G, μ_1, μ_2 are the elastic characteristics of the material; the subscript α (β) denotes differentiation with respect to α (or β).

The equilibrium differential equations of an isotropic, homogeneous, shallow constant-thickness shell in displacements are found in [1], hence, the differential equations corresponding to these conditions are not presented here.

The following geometric conditions

$$w|_{\gamma_1}=0, \qquad \left.\frac{\partial w}{\partial n}\right|_{\gamma_2}=0, \qquad u|_{\gamma_3}=0, \qquad v|_{\gamma_4}=0 \tag{1.2}$$

are given on the sections γ_i $(i=1,\ 2,\ 3,\ 4)$ of the boundary Γ of the domain Ω, where the intersection $\gamma_1\cap\gamma_2$ contains some arc γ_1^0.

The transverse force $N_3(s)$, the moment $\mathbf{M}^*=(M_1^*, M_2^*)$, and the elastic support reaction act on the part $\gamma_5=\Gamma\setminus\gamma_1\cap\gamma_2$ of the contour so that the total transverse force $Q_n(s)$ is determined on $\Gamma\setminus\gamma_1$, and the bending moment $M_n(s)$ on $\Gamma\setminus\gamma_2$

$$Q_n(s)=N_3+\frac{\partial}{\partial s}\left[M_1^*\cos(\beta,n)+M_2^*\cos(\alpha,n)\right]-a_{11}w-a_{12}\frac{\partial w}{\partial n} \tag{1.3}$$
$$M_n(s)=M_1^*\cos(\alpha,n)+M_2^*\cos(\beta,n)-a_{21}w-a_{22}\frac{\partial w}{\partial n}$$

The tangential stress resultant T_1^* is given on the part $\Gamma\setminus\gamma_3$ of the contour, and the tangential stress resultant T_2^* on the part $\Gamma\setminus\gamma_4$ of the boundary

$$T_1^*(s)=N_1(s)-b_{11}u-b_{12}v, \qquad T_2^*(s)=N_2(s)-b_{21}u-b_{22}v \tag{1.4}$$

where N_1, N_2 are external tangential stress resultants, $\|b_{ij}\|$ is a matrix characterizing the elastic support reaction.

2. Functional spaces. The following Conditions are henceforth considered satisfied:

1) The domain Ω is a bounded domain, the finite sum of star domains Ω_k.

2) The boundary Γ of the domain Ω consists of a finite number of closed contours of the Liapunov class $\text{Л}_1(m,0)$ (*).

3) The coefficients A, B of the first quadratic form of the surface S^*, and their derivatives A_α, A_β, B_α, B_β belong to the space $L^\infty(\Omega)$, and

$$A\geqslant m_1, \qquad B\geqslant m_1>0 \tag{2.1}$$

almost everywhere in Ω; m_1 is some constant.

4) The curvatures of the middle surface R_1^{-1}, R_2^{-1}, R_{12}^{-1} belong to the space $L^2(\Omega)$.

5) h, E_1, E_2, G, μ_1, μ_2 belong to the space $L^\infty(\Omega)$, and the inequalities

$$0<m_2\leqslant h, E_1, E_2, \qquad G<\infty, \qquad 0\leqslant\mu_1, \qquad \mu_2\leqslant m_3<1 \tag{2.2}$$

are satisfied almost everywhere in Ω; m_2, m_3 are certain constants.

6) The matrix $\|b_{ij}(s)\|$ is uniformly positive definite in Γ, and the functions $a_{ij}(s)$, $b_{ij}(s)$ are piecewise continuous.

1. The Space $H_1(\Omega)$. The scalar product

$$(w^{(1)}\cdot w^{(2)})_{H_1(\Omega)}=\int_\Omega\left[\left(D_1\varkappa_1^{(1)}+D_{12}\varkappa_2^{(1)}\right)\varkappa_1^{(2)}+\left(D_{21}\varkappa_1^{(1)}+D_2\varkappa_2^{(1)}\right)\varkappa_2^{(2)}+\right.$$
$$\left.+\,2D_k\tau^{(1)}\tau^{(2)}\right]AB\,d\alpha\, d\beta \tag{2.3}$$

is introduced in the set C_1 of functions $w\in C^{(2)}(\Omega)$ and satisfy the first two boundary Conditions in (1.2). The closure of the set C_1 in the appropriate norm (2.3) is

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*) Editor's note. The symbol Л (Cyrillic L) stands for Liapunov.

called the space $H_1(\Omega)$.

2. The Space $H_2(\Omega, \gamma)$. Let C_2 be the set of pairs of functions $\omega^*(u, v) \in C^{(1)}(\Omega)$ which satisfy the two last boundary Conditions in (1.2). The scalar product

$$(\omega^{*(1)} \cdot \omega^{*(2)})_{H_2(\Omega, \gamma)} = \int_\Omega [(E_{11}\varepsilon_{10}^{(1)} + E_{12}\varepsilon_{20}^{(1)})\varepsilon_{10}^{(2)} + (E_{21}\varepsilon_{10}^{(1)} + E_{22}\varepsilon_{20}^{(1)})\varepsilon_{20}^{(2)} + G_1\varepsilon_{30}^{(1)}\varepsilon_{30}^{(2)}] AB d\alpha d\beta + \int_\gamma [(b_{11}u^{(1)} + b_{12}v^{(1)})u^{(2)} + (b_{21}u^{(1)} + b_{22}v^{(1)})v^{(2)}] ds \tag{2.4}$$

$$\varepsilon_{10}^{(i)} = A^{-1}u_\alpha^{(i)} + A_\beta v^{(i)}(AB)^{-1}, \qquad \varepsilon_{20}^{(i)} = B^{-1}v_\beta^{(i)} + B_\alpha u^{(i)}(AB)^{-1}$$

$$\varepsilon_{30}^{(i)} = \frac{A}{B}\left(\frac{u^{(i)}}{A}\right)_\beta + \frac{B}{A}\left(\frac{v^{(i)}}{B}\right)_\alpha, \qquad i = 1, 2$$

is introduced in the set C_2, where γ is some part of the contour Γ. The closure of the set C_2 in the appropriate norm (2.4) is called the space $H_2(\Omega, \gamma)$.

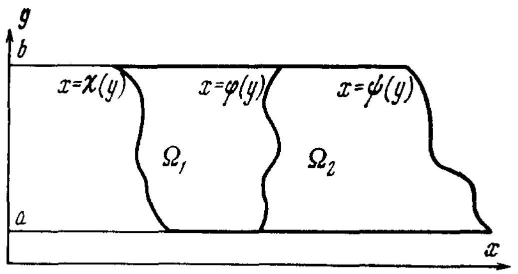

Fig. 1

3. The Space $H_3(\Omega)$. The space $H_3(\Omega)$ is the Hilbert space of vector functions $\omega(u, v, w)$ such that $\omega^*(u, v) \in H_2(\Omega, \gamma)$; $w \in H_1(\Omega)$ with the naturally introduced scalar product. $H_3(\Omega) = H_2(\Omega, \gamma) \times H_1(\Omega)$.

4. The Space $H_4(\Omega, \gamma)$. The space $H_4(\Omega, \gamma)$ is a particular case of the space $H_2(\Omega, \gamma)$: $b_{11} = b_{22} = 1$, $b_{12} = b_{21} = 0$.

5. The Space $H_5(\Omega, \gamma)$. The closure of the set of vector functions $\omega^*(u,v) \in C^{(1)}(\Omega)$ in the norm

$$\|\omega^*\|^2_{H_5(\Omega,\gamma)} = \int_\Omega \left[u_\alpha^2 + v_\beta^2 + \frac{1}{2}(u_\beta + v_\alpha)^2\right] d\alpha d\beta + \int_\gamma (u^2 + v^2) ds \tag{2.5}$$

is the space $H_5(\Omega, \gamma)$. Further, we will examine the properties of the spaces introduced. The domains Ω_1, Ω_2 are adjacent parts of the strip between the lines $y = a$ and $y = b$, as is indicated in Fig. 1. The domain Ω_1 is described by the relation $\chi(y) \leqslant x \leqslant \varphi(y)$, and Ω_2 by the relation $\varphi(y) \leqslant x \leqslant \psi(y)$. The functions $\chi(y)$, $\varphi(y)$, $\psi(y)$ are piecewise continuous in $[a, b]$. Let $c = \inf|\varphi(y) - \chi(y)|$, $c_1 = \sup|\varphi(y) - \chi(y)|$, $d = \sup|\psi(y) - \chi(y)|$, $d_1 = \sup|\psi(y) - \varphi(y)|$ in $[a, b]$.

Lemma 2.1. The inequalities

$$c\|u\|^2_{L^2(\Omega_1\cup\Omega_2)} \leqslant 2d\,[\|u\|^2_{L^2(\Omega_1)} + c_1 d\|u_x\|^2_{L^2(\Omega_1\cup\Omega_2)}] \tag{2.6}$$

$$\|v\|^2_{L^2(\Omega_2)} \leqslant 2d_1\left[\int_{x=\varphi(y)} v^2 ds + d_1\|v_x\|^2_{L^2(\Omega_2)}\right] \tag{2.7}$$

are valid for the functions $u \in W_2^{(1,0)}(\Omega_1 \cup \Omega_2)$, $v \in W_2^{(1,0)}(\Omega_2)$. It is sufficient to carry out the proof for the functions $u \in C^{(1)}(\Omega_1 \cup \Omega_2)$, $v \in C^{(1)}(\Omega_2)$. The corresponding verifications for the functions from the space $W_2^{(1,0)}(\Omega)$ are obtained by passing to the limit. The representation

$$u(x, y) = u(x_0, y) + \int_{x_0}^{x} u_t(t, y)\, dt$$

is valid for the function $u \in C^{(1)}(\Omega_1 \cup \Omega_2)$. After squaring both sides of this expression and estimating the right side by using the Hölder inequality, we obtain

$$u^2(x, y) \leqslant 2u^2(x_0, y) + 2d \int_{\chi(y)}^{\psi(y)} u_t^2\, dt$$

Successive integration with respect to the variable x_0 between $\chi(y)$ and $\varphi(y)$, and then over the domain $\Omega_1 \cup \Omega_2$ (in the variables x, y) results in the inequality

$$c\int_a^b \int_{\chi(y)}^{\psi(y)} u^2 dx dy \leqslant \int_a^b \int_{\chi(y)}^{\psi(y)} [\varphi(y) - \chi(y)]\, u^2 dx\, dy \leqslant$$

$$\leqslant 2d \left\{ \int_a^b \int_{\chi(y)}^{\varphi(y)} u^2 dx\, dy + 2cd \int_a^b \int_{\chi(y)}^{\psi(y)} \left[\int_{\chi(y)}^{\psi(y)} u_t^2\, dt \right] dx\, dy \right\}$$

Hence (2.6) indeed follows. A new function

$$u(x, y) = \begin{cases} v(x, y), & x \geqslant \varphi(y) \\ v(\varphi(y), y), & x \leqslant \varphi(y) \end{cases}$$

is constructed to obtain the estimate (2.7). Taking into account that $u_x = 0$ in the domain $\Omega_1 = \{x, y : \varphi(y) - 1 < x < \varphi(y)\}$ and modifying the proof presented above insignificantly, we can obtain the inequality

$$\int_a^b \int_{\varphi(y)}^{\psi(y)} u^2 dx\, dy \leqslant 2d_1 \left[\int_a^b \int_{\varphi(y)-1}^{\varphi(y)} u^2 dx\, dy + d_1 \| u_x \|^2_{L^2(\Omega_2)} \right]$$

The equality

$$\int_a^b \int_{\varphi(y)-1}^{\varphi(y)} u^2 dx\, dy = \int_{x=\varphi(y)} u^2 ds$$

follows from the definition of the function u, which completes the proof of Lemma 2.1.

Note. If $u(x, y) = 0$ on the contour $x = \varphi(y)$, then the inequality

$$\| u \|_{L^2(\Omega_2)} \leqslant \sqrt{2}\, d_1 \| u_x \|_{L^2(\Omega_2)} \tag{2.8}$$

is valid.

Lemma 2.2. Let γ be a part of the boundary Γ of the domain Ω which contains a certain connecting arc of length $2l > 0$, and furthermore, let there be the vector function $\omega^* \in H_5(\Omega, \gamma)$. Then $\omega^* \in W_2^{(1)}(\Omega)$, and the relationship

$$0 < m \leqslant \frac{\| \omega^* \|_{H_5(\Omega, \gamma)}}{\| \omega^* \|_{W_2^{(1)}(\Omega)}} \leqslant M \tag{2.9}$$

holds with constants m, M independent of the choice of the element ω^*

In the main, the proof of Lemma 2.2 agrees with the proof of the Korn inequality [2]. A slight addition permits proving Lemma 2.2 (and the Korn inequality itself) in the case of the boundary $\Gamma \in Л_1(m, 0)$.

Further, the covering of Ω by domains Δ_k of a special kind will be constructed, and some inequalities will be established for the elements of the spaces $W_2^{(1)}(\Delta_k) \times W_2^{(1)}(\Delta_k)$.

Let $\gamma = AB$ (Fig. 2) be a connecting arc of length $2l > 0$ of the contour Γ which also belongs to the boundary of the domain Ω_1 (see Condition 1). Parallel lines $OC \parallel AF \parallel BM$ are drawn through the points A, B and the middle C of the chord AB, where O is the center of a circle of the star Ω_1 (of radius R_1).

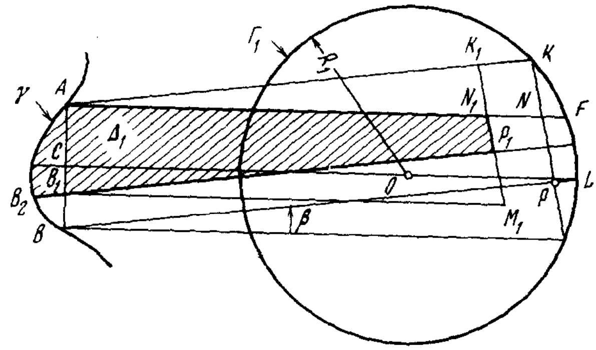

Fig. 2

Let $2l < R_1$. Otherwise, part of the arc γ of such length can be taken. The line $AK \parallel BL$, where L is the point of intersection of the line OC and the circle Γ_1 bounding the starry part of the domain Ω_1. The points N, p are points of intersection of the line KM and the lines AF and BL, respectively. A computation shows that the angle $\beta^* = \angle LBM$ is such that $\sin\beta^* > 8^{-1}lR_1^2D^{-3}$, where D is the diameter of the domain Ω_1. The quadrangle $AKMB$ decreases so that the new quadrangle $AK_1M_1B_1$ is similar to $AKMB$ and the angle $\angle BAK$ is fixed. Let B_2 be the point of intersection between P_1B_1 and the boundary γ, which can lie either outside or inside the quadrangle $AK_1M_1B_1$; Δ_1 is defined as a domain bounded by the part $AB_2 = \sigma_1$ of the arc γ and the broken line $AN_1P_1B_2$ Let δ_1 be the diameter of the domain Δ_1.

Lemma 2.3. The inequality

$$\int_{\Delta_1} (u^2+v^2)\,d\alpha d\beta \leqslant 2^8\delta_1 D^6 l^{-2} R_1^{-4}\Big\{2\int_{\sigma_1}(u^2+v^2)\,ds + \\ + \delta_1\int_{\Delta_1}[2u_\alpha^2+2v_\beta^2+(u_\beta+v_\alpha)^2]\,d\alpha d\beta\Big\} \qquad (2.10)$$

is valid for the vector function $\omega^* \in W_2^{(1)}(\Delta_1)$, where l, D, R_1, σ_1 have been defined above.

Two coordinate systems x_1Oy_1 and x_2Oy_2 such that the Ox_1 axis is parallel to the line AN, and the axis Ox_2 is parallel to BL, are introduced for the proof. It is seen that the domain Δ_1 possesses properties of the domain Ω in Lemma 2.1 in both coordinate systems.

Let u_1, v_1, u_2, v_2 be projections of the vector function $\omega^*(u, v)$ on the Ox_1, Oy_1, Ox_2, Oy_2 axes, respectively. It can be shown that

$$2^{-1} \leqslant \frac{u^2+v^2}{u_1^2+u_2^2} \leqslant \frac{128D^6}{l^2R_1^4} \qquad (2.11)$$

$$2u_\alpha{}^2 + 2v_\beta{}^2 + (u_\beta + v_\alpha)^2 = 2u_{ix_i}^2 + 2v_{iy_i}^2 + (u_{iy_i} + v_{ix_i})^2 \quad (i = 1,\ 2) \tag{2.12}$$

Application of Lemma 2.1 and relations (2.11), (2.12) result in the following:

$$\int\limits_{\Delta_1} (u^2 + v^2)\, d\alpha\, d\beta \leqslant 2^8\delta_1 D^8 l^{-2} R_1^{-4} \left\{ \int\limits_{\sigma_1} (u_1^2 + u_2^2)\, ds + \delta_1 \left[\int\limits_{\Delta_1} u_{1x_1}^2 dx_1 dy_1 + \right.\right.$$

$$\left.\left. + \int\limits_{\Delta_1} u_{2x_2}^2 dx_2 dy_2 \right]\right\} \leqslant 2^8\delta_1 D^6 l^{-2} R_1^{-4} \left\{ 2 \int\limits_{\sigma_1} (u^2 + v^2) ds + \delta_1 \int\limits_{\Delta_1} \left[2u_\alpha{}^2 + 2v_\beta{}^2 + (u_\beta + v_\alpha)^2\right] l\alpha\, d\beta \right\}$$

which proves Lemma 2.3.

Lemma 2.4. The domain Ω can be covered by a finite system of domains Δ_i $(i = 1, 2, \ldots, N(\delta))$, each of whose diameters is less than an arbitrarily assigned number $\delta > 0$ with the following Properties:

1) Δ_1 is the domain constructed above;

2) Each of the domains Δ_i $(i = 1, 2, \ldots, N(\delta))$ is a starry domain relative to each point of its certain interior circle K_i of the radius ρ_i and

$$K_i \subset \bigcup_{j=1}^{i-1} \Delta_j$$

3) For all $\delta > 0$ the inequalities

$$\frac{\operatorname{diam} \Delta_i}{\rho_i} \leqslant \frac{10D}{R_0}, \qquad i = 2, 3, \ldots, N(\delta) \tag{2.13}$$

hold.

Indeed, a circle in Ω of diameter less than δ with center at the middle of the segment N_1P_1 (Fig. 2) can be considered the domain Δ_2. The same circle, but with center on the boundary of the domain $\Delta_1 \cup \Delta_2$ is Δ_3, etc. A finite number of such circles can cover any strictly interior sub-domain of Ω. It is seen that this part of the covering satisfies the conditions of the lemma. There remains to construct a covering of the boundary strip Ω_ε of width ε of the domain Ω. It is evidently sufficient to construct it for any of the domains Ω_k, for example Ω_1. A circle Γ_2 of radius $2^{-1} R_1$ is constructed in the domain Ω_1, to be concentric with the circle Γ_1 bounding the starry part of the domain Ω_1 (Fig. 3). Two tangents CL and CM to the circle Γ_2, between which the angle is $2\alpha_0$, are drawn from an arbitrary point C of the boundary of Ω_1. Evidently $\sin \alpha_0 \geqslant 2^{-1} R_1 D^{-1}$. An angle FNK, whose sides FN and KN are a distance ε from the sides LC and MC, respectively, is constructed within the angle LCM. A circle Γ_3 of radius ε with center O is inscribed in the angle FNK. The domain bounded by the semicircle APA_1, the tangents to the circle $AB \| A_1B_1 \| OC$ and the part BCB_1 of the boundary Ω_1 is called Δ_c. It is seen that for sufficiently small ε a part of the strip Ω_ε included between the lines AB and A_1B_1 and the corresponding neighborhood of the arc BCB_1 does not intersect the circle Γ_3, i.e. the Property (2) is satisfied. A direct computation shows that $\operatorname{diam} \Delta_c\varepsilon^{-1} < 10\, DR_1^{-1}$, i.e. Property (3) is also satisfied. The construction described

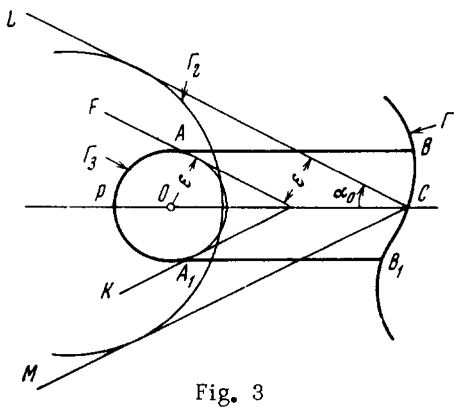

Fig. 3

above is performed at each point of the boundary of Ω , and then by using the Bolzano lemma a finite covering is selected. This completes the proof.

Lemma 2.5. Let Δ be a starry domain relative to the circle K of radius R , with a boundary of the class $Л_1(m, 0)$ (see Footnote p. 653), whose diameter is δ. Then for the vector-function $\omega^*(u, v) \in W_2^{(1)}(\Delta)$, the inequality

$$\int_\Delta (u^2+v^2)\, d\alpha d\beta \leqslant 11\cdot 2^{-1}\delta^4 R^{-4}\Big\{\int_K (u^2+v^2)\, d\alpha d\beta + 3\delta^2 \int_\Delta [2u_\alpha^2 + 2v_\beta^2 + (u_\beta+v_\alpha)^2]\, d\alpha d\beta\Big\} \tag{2.14}$$

is valid.

For the proof, a circle O_1 of radius δ, concentric with the circle K , (Fig. 4) is drawn. The sides A_1B_1 and C_1D_1 of some square $A_1B_1C_1D_1$ inscribed in the circle K, are continued until they intersect the circle O_1 at the points F_1, N_1. The domain bounded by the lines A_1L, C_1M, by the arc $A_1A_2C_1$ and part of the boundary LM, is denoted by T_1. The figure $A_1F_1N_1C_1$ is rotated through a certain angle α^* relative to the point O such that the point N_1 is displaced to the middle of the arc F_1N_1. The appropriate part of the domain Δ is denoted by T_2. Moreover, the figure $A_1F_1N_1C_1$ is rotated through the angles $2\alpha^*, 3\alpha^*, \ldots, n\alpha^*$ relative to the point O , and their corresponding parts of Δ are denoted by $T_3, T_4, \ldots, T_{n+1}$, where n is selected as the least from the condition $(n+1)\alpha^* \geqslant 2\pi$. It can be considered that $\sin\alpha^* = \sqrt{2}R\delta^{-1}$, and therefore $\alpha^* > \sqrt{2}\, R\delta^{-1}$, from which $n \leqslant \sqrt{2}\pi\delta R^{-1}$. A local x_iOy_i coordinate system is introduced in each of the domains T_i .

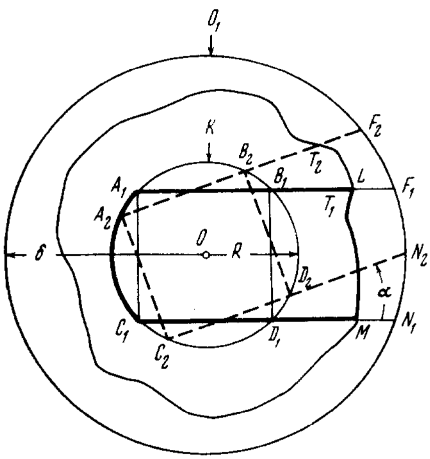

Fig. 4

Let u_i be the projection of the vector $\omega^*(u, v)$ on the Ox_i axis which goes along ON_{i+1}. By virtue of Lemma 2.1, the inequality

$$\int_{T_i} u_i^2 dx_i dy_i \leqslant \sqrt{2}\,(1+\delta R^{-1})\Big\{\int_{T_i\cap K} u_i^2 dx_i dy_i + 2R(R+\delta)\int_{T_i} u_{ix_i}^2 dx_i dy_i\Big\} \tag{2.15}$$

$$\sum_{i=1}^{n+1}\int_{T_i} u_i^2 dx_i dy_i \leqslant \sqrt{2}\,(1+\delta R^{-1})(n+1)\Big\{\int_K (u^2+v^2)\, d\alpha d\beta + R(R+\delta)\int_\Delta [2u_\alpha^2 + 2v_\beta^2 + (u_\beta+v_\alpha)^2]\, d\alpha\, d\beta\Big\} \tag{2.16}$$

holds. Evidently

$$\bigcup_{i=1}^{n}(T_i \cap T_{i+1}) = \Delta$$

The inequality

$$u^2 + v^2 \leqslant 2(\sin\alpha^*)^{-2}(u_i^2 + u_{i+1}^2) \tag{2.17}$$

holds in each of the domains $T_i \cap T_{i+1}$. Hence

$$\int_\Delta (u^2+v^2)\,d\alpha\,d\beta \leqslant \sum_{i=1}^{n} \int_{T_i \cap T_{i+1}} (u^2+v^2)\,d\alpha\,d\beta \leqslant 2(\sin\alpha^*)^{-2} \sum_{i=1}^{n} \Big[\int_{T_i \cap T_{i+1}} u_i^2 dx_i dy_i + $$
$$+ \int_{T_i \cap T_{i+1}} u_{i+1}^2\,dx_i dy_i \Big] \leqslant 4(\sin\alpha^*)^{-2} \sum_{i=1}^{n+1} \int_{T_i} u_i^2 dx_i dy_i \tag{2.18}$$

The inequality

$$\int_\Delta (u^2+v^2)\,d\alpha d\beta \leqslant 4(n+1)(1+\delta R^{-1})(\sin\alpha^*)^{-2} \Big\{ \int_K (u^2+v^2)\,d\alpha\,d\beta + $$
$$+ R(R+\delta) \int_\Delta [2u_\alpha^2 + 2v_\beta^2 + (u_\beta + v_\alpha)^2]\,d\alpha\,d\beta \Big\}$$

follows from the inequalities (2.16), (2.18), from which the estimate (2.14) results.

Lemma 2.6. Let γ, a part of the boundary of the domain Ω, contain a connecting arc of length $2l > 0$. Furthermore, let the vector function be $\omega^* \in H_4(\Omega, \gamma)$. In this case $\omega^* \in H_5(\Omega, \gamma)$, where the relationship

$$0 < m \leqslant \frac{\|\omega^*\|_{H_4(\Omega,\gamma)}}{\|\omega^*\|_{H_5(\Omega,\gamma)}} \leqslant M \tag{2.19}$$

with constants m, M independent of the choice of ω^*, holds.

The right side of the inequality (2.19) is proved by using Lemma 2.2. The proof of the left inequality is by contradiction. Let the inequality $m\|\omega^*\|_{H_5(\Omega,\gamma)} \leqslant \|\omega^*\|_{H_4(\Omega,\gamma)}$ not be satisfied. In this case, there is a sequence $\omega_n^* \in C^{(1)}(\Omega)$ such that

$$\|\omega_n^*\|_{H_5(\Omega,\gamma)} = 1, \quad \|\omega_n^*\|_{H_4(\Omega,\gamma)} \to 0 \ \text{ when } n \to \infty \tag{2.20}$$

It can be considered that ω_n^* converges weakly to ω_0^* in $H_5(\Omega, \gamma)$. It follows from the relation (2.20) that the sequences

$$\varepsilon_{10n} = A^{-1}u_{n\alpha} + A_\beta v_n (AB)^{-1}, \qquad \varepsilon_{20n} = B^{-1}v_{n\beta} + B_\alpha u (AB)^{-1}$$
$$\varepsilon_{30n} = \frac{A}{B}\left(\frac{u_n}{A}\right)_\beta + \frac{B}{A}\left(\frac{v_n}{B}\right)_\alpha \tag{2.21}$$

converge strongly to zero in the space $L^2(\Omega)$ and

$$\lim_{n\to\infty} \int_\gamma (u_n^2 + v_n^2)\,ds = 0$$

A new function $\theta(\varphi, \psi) = (B^{-1}u, \ A^{-1}v)$ is introduced. From the fact that $\|\theta_n\|_{H_4(\Omega,\gamma)} \to 0$ as $n \to \infty$ there follows $\|\omega_n^*\|_{H_4(\Omega,\gamma)} \to 0$ as $n \to \infty$, and conversely. The inequality

$$2\varphi_{n\alpha}^2 + 2\psi_{n\beta}^2 + (\varphi_{n\beta} + \psi_{n\alpha})^2 \leqslant 2c_1(\varepsilon_{10n}^2 + \varepsilon_{20n}^2 + \varepsilon_{30n}^2) + 2c_2(\varphi_n^2 + \psi_n^2) \tag{2.22}$$

is valid for the function θ_n almost everywhere in Ω, where the constants c_1, c_2 depend only on the estimate of the coefficients of the first quadratic form A, B, and are finite by virtue of Condition (3).

Having constructed the covering of domain Ω (Lemma 2.4) and applied the Lemma 2.3, we can obtain the inequality

$$\int_{\Delta_1} [2\varphi_{n\alpha}^2 + 2\psi_{n\beta}^2 + (\varphi_{n\beta} + \psi_{n\alpha})^2]\,d\alpha\,d\beta \leqslant 2c_1 \int_\Omega (\varepsilon_{10n}^2 + \varepsilon_{20n}^2 + \varepsilon_{30n}^2)\,d\alpha\,d\beta +$$

$$+64c_2\delta_1 D^6 l^{-2} R_1^{-4}\left\{4\int\limits_{\gamma}(\varphi_n{}^2+\psi_n{}^2)\,ds+\delta_1\int\limits_{\Delta_1}[2\varphi_{n\alpha}^2+2\psi_{n\beta}^2+(\varphi_{n\beta}+\psi_{n\alpha})^2]\,d\alpha d\beta\right\} \quad (2.23)$$

If follows from the inequality (2.23) that

$$\int\limits_{\Delta_1}[2\varphi_{n\alpha}^2+2\psi_{n\beta}^2+(\varphi_{n\beta}+\psi_{n\alpha})^2]\,d\alpha d\beta\leqslant 4c_1\int\limits_{\Delta_1}(\varepsilon_{10n}^2+\varepsilon_{20n}^2+\varepsilon_{30n}^2)\,d\alpha d\beta+$$
$$+2^{11}c_2\delta_1 D^6 l^{-2}R_1^{-4}\int\limits_{\gamma}(\varphi_n{}^2+\psi_n{}^2)\,ds \quad (2.24)$$

where $\delta_1{}^2=2^{-8}l^2R_1{}^4D^{-6}c_2{}^{-1}$. From here and from Lemma 2.2 it follows that $\theta_n\to 0$ in $W_2{}^1(\Delta_1)$, and therefore $\|\vartheta_n\|_{L^2(\Delta_1)}\to 0$ as $n\to\infty$. The inequalities

$$\int\limits_{\Delta_i}[2\varphi_{n\alpha}^2+2\psi_{n\beta}^2+(\varphi_{n\beta}+\psi_{n\alpha})^2]\,d\alpha d\beta\leqslant 4\int\limits_{\Delta_i}(\varepsilon_{10n}^2+\varepsilon_{20n}^2+\varepsilon_{30n}^2)\,d\alpha d\beta+$$
$$+88c_2(10D)^4R_0{}^{-4}\int\limits_{K_i}(\varphi_n{}^2+\psi_n{}^2)\,d\alpha d\beta \quad (2.25)$$

are valid in the domains Δ_i for $\delta^2\leqslant 66^{-1}(10D)^{-4}R_0{}^4c_2{}^{-1}$.

Considering the inequality (2.25) successively in the domains $\Delta_2, \Delta_3,\ldots,\Delta_{N(\delta)}$ and taking account of the properties of the covering, it can be concluded that $\|\theta_n\|_{H_2(\Omega,\gamma)}\to 0$ as $n\to\infty$, which indeed proves the lemma.

Lemma 2.7. Let $\omega^*(u,v)\in H_2(\Omega,\gamma)$, where γ is defined as in Lemma 2.6. In this case $\omega^*\in W_2{}^{(1)}(\Omega)$ and

$$0<m\leqslant\frac{\|\omega^*\|_{H_2(\Omega,\gamma)}}{\|\omega^*\|_{W_2^{(1)}(\Omega)}}\leqslant M \quad (2.26)$$

with constants m, M independent of the choice of ω^*.

Lemma 2.8. Let the function be $w\in H_1(\Omega)$, then $w\in W_2{}^{(2)}(\Omega)$ and

$$0<m\leqslant\frac{\|\omega\|_{H_1(\Omega)}}{\|w\|_{W_2^{(2)}(\Omega)}}\leqslant M \quad (2.27)$$

with constants m, M independent of the choice of the element w.

New functions $u=A^{-1}w_\alpha$, $v=B^{-1}w_\beta$ are introduced for the proof. Such a substitution reduces Lemma 2.8 to Lemma 2.6.

Lemmas 2.6 – 2.8 mean that the appropriate imbedding theorems presented in [3], which will be used, are valid in the spaces $H_1(\Omega)$, $H_2(\Omega,\gamma)$.

3. Generalized solution and solvability of the problem. Let the following conditions

$$Z\in H_{-1}(\Omega),\quad X\in L^p(\Omega),\quad Y\in L^p(\Omega),\quad M_1{}^*\in L^p(\Gamma) \quad (3.1)$$
$$M_2{}^*\in L^p(\Gamma),\quad N_1\in L^p(\Gamma),\quad N_2\in L^p(\Gamma),\quad N_3\in L(\Gamma)\qquad (p>1)$$

be satisfied, where $H_{-1}(\Omega)$ is a negative space with the norm

$$\|Z\|_{H_{-1}(\Omega)}=\sup\left\{\left|\int\limits_{\Omega}Zw\,d\alpha d\beta\right|\|w\|_{H_1(\Omega)}^{-1}\right\}$$

In particular $L^p(\Omega)\subset H_{-1}(\Omega)$, $p\geqslant 1$.

At the points of Γ where the stress resultants N_i are moments $M_i{}^*$ are not specified, these quantities are predefined zero.

It is known that the shell equilibrium condition can be expressed by using the Lagrange principle of virtual displacements, which is in this case

$$(\omega\cdot\mathbf{a})_{H_3(\Omega)} = -\int_{\Omega}\Big\{T_1(R_1^{-1}\chi + w_\alpha\chi_\alpha A^{-2}) + T_2(R_2^{-1}\chi + w_\beta\chi_\beta B^{-2}) -$$
$$- S\,[2R_{12}^{-1}\chi - (AB)^{-1}(w_\alpha\chi_\beta + w_\beta\chi_\alpha)] + \Big[E_{11}\Big(R_1^{-1}w + \frac{1}{2}A^{-2}w_\alpha^2\Big) +$$
$$+ E_{12}\Big(R_2^{-1}w + \frac{1}{2}B^{-2}w_\beta^2\Big)\Big]\,[A^{-1}\varphi_\alpha + A_\beta\psi(AB)^{-1}] + \Big[E_{21}\Big(R_1^{-1}w +$$
$$+ \frac{1}{2}A^{-2}w_\alpha^2\Big) + E_{22}\Big(R_2^{-1}w + \frac{1}{2}B^{-2}w_\beta^2\Big)\Big]\,[B^{-1}\psi_\beta + B_\alpha\varphi(AB)^{-1}] +$$
$$+ G_1\Big(\frac{w_\alpha w_\beta}{AB} - \frac{2w}{R_{12}}\Big)\Big[\frac{A}{B}\Big(\frac{\varphi}{A}\Big)_\beta + \frac{B}{A}\Big(\frac{\psi}{B}\Big)_\alpha\Big]\Big\}\,AB\,d\alpha d\beta +$$
$$+ \int_{\Omega}[(Z - Xw_\alpha A^{-1} - Yw_\beta B^{-1})\chi + X\varphi + Y\psi]\,AB\,d\alpha d\beta +$$
$$+ \int_{\Gamma}\Big[N_1\varphi + N_2\psi + N_3\chi + A^{-1}M_1{}^*\chi_\alpha + B^{-1}M_2{}^*\chi_\beta - \Big(a_{11}w + a_{12}\frac{\partial w}{\partial n}\Big)\chi -$$
$$- \Big(a_{21}w + a_{22}\frac{\partial w}{\partial n}\Big)\frac{\partial\chi}{\partial n}\Big]ds \qquad (3.2)$$

The vector function $\mathbf{a}\ (\varphi, \psi, \chi)$ is the virtual displacement here.

D e f i n i t i o n 3.1. The vector function $\omega\ (u, v, w) \in H_3\ (\Omega)$, turning the integral equality (3.2) into an identity for any vector function $\mathbf{a}\,(\varphi, \psi, \chi) \in H_3\ (\Omega)$, is called a generalized solution of the shell equilibrium problem.

Estimating each member in (3.2) by using the Hölder inequality, it can be seen that under the conditions imposed on the shell parameters, the external forces and the vector functions ω, $\mathbf{a}$, the inequality (3.2) is meaningful. Considering the right side of (3.2) separately for fixed $\omega\ (u, v, w) \in H_3\ (\Omega)$ as a functional in the vector function $\mathbf{a}\ (\varphi, \psi, \chi) \in H_3\ (\Omega)$, it is verified directly that it is linear and continuous in the space $H_3\ (\Omega)$, and therefore, we represent it as a scalar product in $H_3\ (\Omega)$ by the Riesz theorem. This representation defines some nonlinear operator $K\omega$ in the space $H_3(\Omega)$, and (3.2) itself becomes

$$(\omega\cdot\mathbf{a})_{H_3(\Omega)} = (K\omega\cdot\mathbf{a})_{H_3(\Omega)} \qquad (3.3)$$

Therefore, finding the generalized solution of the problem is equivalent to solving an operator equation in the space $H_3\ (\Omega)$

$$\omega = K\omega \qquad (3.4)$$

L e m m a 3.1. The operator K is strongly continuous in the space $H_3\ (\Omega)$. The assertion in the lemma is verified directly by using the Hölder inequality and the Sobolev-Kondrashev theorem on the complete continuity of the imbedding operator in the spaces $W_p{}^{(l)}\ (\Omega)$ [3].

The Schauder-Leray principle on the fixed point of an operator is used to prove the solvability of (3.4). Let $S\ (1, 0)$ be a sphere of unit radius in the space $H_3\ (\Omega)$ with center at zero: $\|\,\omega\,\|_{H_3\ (\Omega)} = 1$. The projections of the sphere $S\ (1, 0)$ using the mapping

$$w = Rw_1, \qquad \omega^* = h^*R^2\omega_1{}^* \qquad (3.5)$$

where $R > 0$, $h^* > 0$ are certain constants, defines an "ellipsoid" $C\ (h^*, R, 0)$ in the space $H_3\ (\Omega)$. For fixed constants $h^* > 1$, $R > 1$ the ellipsoid is the boundary of some connected convex domain containing the unit sphere with center at zero of the space

$H_3(\Omega)$.

By the Schauder-Leray principle, for the solvability of (3.4) it is sufficient to prove that the completely continuous vector field is homotopic in some ellipsoid $C(h^*, R, 0)$ to a completely continuous vector field $I\omega$, where I is the identity operator for which rotation [4] equals plus one. For homotopy it is sufficient to show that

$$(I - tK)\,\omega \neq 0 \quad \text{for} \quad \omega \in C(h^*, R, 0),\ t \in [0, 1] \tag{3.6}$$

and some $h^* > 0$, $R > 0$. To prove (3.6), some properties of the functional

$$\Phi(\omega, t) = ((\omega - tK\omega)\cdot \mathrm{a})_{H_3(\Omega)} \tag{3.7}$$

are examined, where the vector function is $\mathrm{a} = (2u, 2v, w)$.

Lemma 3.2. Let the sequence $\omega_k \in H_3(\Omega)$, which converges weakly to $\omega_0 \in H_3(\Omega)$, be such that the sequence

$$\Phi_{41}(\omega_k) \to 0 \quad \text{for} \quad k \to \infty \tag{3.8}$$

Here

$$\Phi_{41}(\omega) = \int_\Omega [(E_{11}\varepsilon_4 + E_{12}\varepsilon_5)\varepsilon_4 + (E_{21}\varepsilon_4 + E_{22}\varepsilon_5)\varepsilon_5 + G_1\varepsilon_6^2]\,AB\,d\alpha d\beta +$$

$$+ h^{*2}\int_\Gamma [(b_{11}u + b_{12}v)u + (b_{21}u + b_{22}v)v]\,ds, \quad 2\varepsilon_4 = 2h^*[A^{-1}u_\alpha +$$

$$+ A_\beta v(AB)^{-1}] + A^{-2}w_\alpha^2, \quad 2\varepsilon_5 = 2h^*[B^{-1}v_\beta + B_\alpha u(AB)^{-1}] + B^{-2}w_\beta^2$$

$$\varepsilon_6 = h^*\left[\frac{A}{B}\left(\frac{u}{A}\right)_\beta + \frac{B}{A}\left(\frac{v}{B}\right)_\alpha\right] + \frac{w_\alpha w_\beta}{AB}, \qquad h^* > 0 \tag{3.9}$$

In this case $\omega_0 = 0$. It follows from (3.8) and (3.9) that

$$\int_\Omega AB\,[\varepsilon_4(\omega_k) + \varepsilon_5(\omega_k)\,d\alpha d\beta \to 0, \qquad k \to \infty \tag{3.10}$$

$$2h^*\int_\Omega [(Bu_k)_\alpha + (Av_k)_\beta]\,d\alpha d\beta + \int_\Omega (BA^{-1}w_{k\alpha}^2 + AB^{-1}w_{k\beta}^2)\,d\alpha d\beta \to 0, \quad k \to \infty \tag{3.11}$$

By virtue of the theorem on the complete continuity of the imbedding of $H_1(\Omega)$ in $W_2^{(1)}(\Omega)$ and the fact that the first integral is a continuous linear function in $H_2(\Omega)$, we can pass to the limit in (3.11)

$$2h^*\int_\Omega [(Bu_0)_\alpha + (Av_0)_\beta]\,d\alpha d\beta + \int_\Omega (BA^{-1}w_{0\alpha}^2 + AB^{-1}w_{0\beta}^2)\,d\alpha d\beta = 0 \tag{3.12}$$

and since

$$\int_\Gamma [(b_{11}u_0 + b_{12}v_0)u_0 + (b_{21}u_0 + b_{22}v_0)v_0]\,ds = 0 \tag{3.13}$$

then

$$\int_\Omega [(Bu_0)_\alpha + (Av_0)_\beta]\,d\alpha d\beta = 0 \tag{3.14}$$

From (3.12) and (3.14) results $w_0 \equiv 0$, hence, from (3.8) and the Sobolev theorem on the complete continuity of the imbedding of $H_1(\Omega)$ into $W_4^{(1)}(\Omega)$ it follows that $\omega_k^* \Rightarrow 0$ as $k \to \infty$ in the space $H_2(\Omega)$, which proves the lemma completely.

Lemma 3.3. For some fixed h^* and for sufficiently large R the inequality

$$\Phi(\omega, t) \geqslant \delta R^2, \qquad \omega \in C(h^*, R, 0) \tag{3.15}$$

holds on the ellipsoid $C\ (h^*,\ R,\ 0)$ where $\sigma > 0$ is some constant.

The proof is separated into three parts.

1. The set $T\ (1,0)$

$$T\ (1,\ 0)=\{\omega:\|\ \omega\|_{H_3(\Omega)}=1,\quad \|\ \omega^*\|_{H_2(\Omega)}\geqslant {}^1/_2\}$$

is considered on the unit sphere $S\ (1,0)$. The functional $\Phi\ (\omega,\ t)$ has the following structure on the ellipsoid $C\ (h^*,\ R,\ 0)$:

$$\Phi\ (\omega,\ t)=\sum_{k=1}^{4} R^k\Phi_k\ (\omega_1,\ t)$$

where ω_1 is the prototype of ω on the unit sphere. Here $\Phi_k\ (\omega,\ t)$ are some "homogeneous" functionals of degree $k=1,2,3,4$. It is seen that all the functionals $\Phi_k\ (\omega_1,\ t)$ are bounded on the unit sphere $\|\ \omega_1\|_{H_3\ (\Omega)}=1$ for all $t\in[0,1]$

$$|\ \Phi_k\ (\omega_1,\ t)\ |\leqslant\sigma_k<\infty,\qquad k=1,\ 2,\ 3,\ 4$$

where $\sigma_k>0$ are certain constants independent of just the shell parameters and the external load.

The functional $\Phi_4(\omega,\ t)$ is considered on the set $T\ (1,0)$:

$$\Phi_4\ (\omega,\ t)=2h^{*2}\|\ \omega^*\ \|^2_{H_2(\Omega)}+th^*\int_\Omega\Big\{(E_{11}w_\alpha{}^2A^{-2}+E_{12}w_\beta{}^2B^{-2})\ [A^{-1}u_\alpha+A_\beta v\ (AB)^{-1}]+$$

$$+\ (E_{21}w_\alpha{}^2A^{-2}+E_{22}w_\beta{}^2B^{-2})\ [B^{-1}v_\beta+B_\alpha u\ (AB)^{-1}]+G_1\ \frac{w_\alpha w_\beta}{AB}\left[\frac{A}{B}\left(\frac{u}{A}\right)_\beta+\right.$$

$$\left.\left.+\frac{B}{A}\left(\frac{v}{B}\right)_\alpha\right]\right\}ABd\alpha d\beta+\frac{1}{2}\ t\int_\Omega\Big[(E_{11}w_\alpha{}^2A^{-2}+E_{12}w_\beta{}^2B^{-2})\ w_\alpha{}^2A^{-2}+$$

$$+\ (E_{21}w_\alpha{}^2A^{-2}+E_{22}w_\beta{}^2B^{-2})\ w_\beta{}^2B^{-2}+4G_1\ \frac{w_\alpha{}^2w_\beta{}^2}{A^2B^2}\Big]\ ABd\alpha d\beta \tag{3.16}$$

It us seen that

$$\Phi_4\ (\omega,\ t)\geqslant h^{*2}-th^*c_3,\qquad \omega\in T\ (1,\ 0) \tag{3.17}$$

where c_3 is a constant bounding the first integral in (3.16) on the unit sphere. The second integral is always nonnegative. Let the constant $h^*>0$ be fixed by the equality

$$h^{*2}-h^*c_3=1 \tag{3.18}$$

Then the inequality

$$\Phi\ (\omega,\ t)\geqslant R^4-\sum_{k=1}^{3}\sigma_kR^k-\sigma_2R^3 \tag{3.19}$$

Is valid in the set $C_1\ (h^*,\ R,\ 0)$, which is the image of $T\ (1,0)$.

2. The functional $\Phi\ (\omega,\ t)$ can be reduced to the form

$$\Phi\ (\omega,\ t)=\|\ w\|^2_{H_1(\Omega)}+2\ (1-t)\ \|\ \omega^*\ \|^2_{H_2(\Omega)}+2t\int_\Omega(T_1\varepsilon_1+T_2\varepsilon_2+S\varepsilon_3)\ ABd\alpha d\beta\ -$$

$$-\ t\int_\Omega(T_1R_1^{-1}+T_2R_2^{-1}-2SR_{12}^{-1})\ wABd\alpha d\beta-t\int_\Omega[2Xu+2Yv+(Z-Xw_\alpha A^{-1}-$$

$$-\ Yw_\beta B^{-1})\ w]\ ABd\alpha d\beta-t\int_\Gamma[2N_1u+2N_2v+N_3w+M_1{}^*w_\alpha A^{-1}+M_2{}^*w_\beta B^{-1}- \tag{3.20}$$

$$-\left(a_{11}w+a_{12}\frac{\partial w}{\partial n}\right)w-\left(a_{21}w+a_{22}\frac{\partial w}{\partial n}\right)\frac{\partial w}{\partial n}-2\ (b_{11}u+b_{12}v)\ u-2\ (b_{21}u+b_{22}v)\ v]\ ds$$

By virtue of the estimate (2.2), almost everywhere in Ω (3.21)

$$|\ (T_1R_1^{-1}+T_2R_2^{-1}-2SR_{12}^{-1})\ w\ |\leqslant{}^1/_2\ (T_1\varepsilon_1+T_2\varepsilon_2+S\varepsilon_3)+c_4w^2\ (R_1^{-2}+R_2^{-2}+R_{12}^{-2})$$

where c_4 is a constant dependent on the constants m_1, m_2, m_3. The set

$$\| w \|^2_{H_1(\Omega)} - 2h^* \left| \int_\Omega (Xu + Yv)\, ABd\alpha d\beta \right| - \left| \int_\Omega (Xw_\alpha A^{-1} + Yw_\beta B^{-1})\, wABd\alpha d\beta \right| -$$
$$- 2h^* \left| \int_\Gamma (N_1 u + N_2 v)\, ds \right| - \left| \int_\Gamma \left[\left(a_{11} w + a_{12} \frac{\partial w}{\partial n} \right) w + \left(a_{21} w + a_{22} \frac{\partial w}{\partial n} \right) \frac{\partial w}{\partial n} \right] ds \right| -$$
$$- c_4 \int_\Omega (R_1^{-2} + R_2^{-2} + R_{12}^{-2})\, ABd\alpha d\beta\, (\max_\Omega w^2) \geqslant 4^{-1} \tag{3.22}$$

is considered on the part of the unit sphere $S_1\,(1,0) = S\,(1,0) \setminus T\,(1,0)$.

The inequality

$$\Phi\,(\omega,\, t) \geqslant 4^{-1} R^2 - c_5 R \tag{3.23}$$

is satisfied on the projection of $S_1\,(1,0)$ onto $C\,(h^*,\, R,\, 0)$.

3. The functional $\Phi\,(\omega, t)$ is considered on the image of the set $S_2(1,0) = S(1,0) \setminus \{T\,(1,0) \cup S_1\,(1,0)\}$. Sequences $\{\omega_k,\, t_k\}$ such that $0 \leqslant t_k \leqslant 1$ and ω_k converges weakly to zero, cannot exist in the set $S_2\,(1,0)$. Indeed, two inequalities are satisfied in $S_2\,(1,\,0)$: an inequality reverse to the inequality (3.22) and

$$\| w \|^2_{H_1(\Omega)} \geqslant {}^3/_4 \tag{3.24}$$

It follows from the first inequality and the Sobolev-Kondrashev theorem on complete continuity of the imbedding operator that

$$\| w_n \|^2_{H_1(\Omega)} \leqslant 4^{-1} + r_n$$

where $r_n \to \infty$ as $n \to \infty$, which contradicts (3.27).

In the set $S_2\,(1,0)$ the form is $\Phi_{41}\,(\omega) \geqslant \lambda > 0$, where λ is some constant. Otherwise, a sequence $\{\omega_k\} \in S_2\,(1,0)$, weakly convergent to ω_0 exists such that

$$\Phi_{41}\,(\omega_k) \to 0 \quad \text{for} \quad k \to \infty$$

It follows from Lemma 3.2 that $\omega_0 = 0$, but this contradicts the first assertion in Sect. 3. Therefore, the estimate

$$\Phi\,(\omega,\, t) \geqslant {}^3/_4 R^2 + t\,(\lambda R^4 - \sigma_3 R^3 - \sigma_5 R^2 - \sigma_1 R) \tag{3.25}$$

is valid on the image of the set $S_2\,(1,0)$ in $C\,(h^*,\, R,\, 0)$.

The assertion of Lemma 3.3 follows from the inequalities (3.19), (3.23), (3.25). Lemma 3.4 results directly from Lemmas 3.1, 3.3.

Lemma 3.4. The rotation of a completely continuous vector field $I - K$ equals plus one on the ellipsoid $C\,(h^*,\, R,\, 0)$ for all sufficiently large R and h^* fixed by (3.18).

Theorem. Let Conditions (1) - (6) of Sect. 2 and (3.1) be satisfied. In this case there exists a generalized solution of the shell equilibrium problem in the sense of the Definition 3.1, where all possible solutions of the problem are found within some sphere of finite radius of the space $H_3\,(\Omega)$.

The theorem follows from the Lemma 3.4 and the Schauder-Leray principle on the fixed point of an operator.

Note. Modifying the proof insignificantly, we obtain a theorem on the existense of a solution of the shell equilibrium problem with the following boundary conditions. Conditions (1.3), (1.4) are retained, and conditions (1.2) are replaced by the following, respectively:

$$w\,|_{\gamma_1} = 0, \qquad M_n\,|_{\gamma_2} = M_1{}^* \cos(\alpha, n) + M_2{}^* \cos(\beta, n) - k\,(s) \frac{\partial w}{\partial n}$$

$$u|_{\gamma_3} = f_1(s), \quad v|_{\gamma_4} = f_2(s), \quad f_1 \in H_{1/2}(\gamma_3), \quad f_2 \in H_{1/2}(\gamma_4)$$

where $k(s) \geqslant k_0 > 0$ is a piecewise-continuous function, k_0 is some constant, and $H_{1/2}(\gamma)$ is a Sobolev-Slobodetskii space.

Note 2. Using the explicit form of the operator K we can obtain sufficient conditions for the uniqueness of the solution, can study the differential properties of the solutions, and can also give a foundation for the Bubnov-Galerkin method.

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ASYMPTOTIC METHOD OF DETERMINING THE CRITICAL BUCKLING LOADS OF SHALLOW STRICTLY CONVEX SHELLS OF REVOLUTION

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An asymptotic method using the presence of a natural small parameter (the relative wall-thinness) is applied to determine the state of stress and strain of shallow strictly convex shells of revolution subjected to an axisymmetric load. In particular, asymptotic values of the upper and lower critical shell buckling loads are deduced under diverse boundary conditions and loading methods. An example of a spherical shell under uniform external pressure is examined. In the case of rigid clamping of the edge, the known result is obtained in [1] for the upper critical pressure. The values found for the upper critical pressures of spherical shells are in good agreement with the results of numerical computations on an electronic computer [2 – 13], and permit their continuation into the domain of arbitrarily thin shells where the machine computation is of low efficiency.

1. On the formulation of the problem. A system of nonlinear differential equations of axisymmetric deformation of shallow shells of revolution is considered [14]